ROLLING OF A RIGID BODY ON A FIXED SURFACE

(O KATANII TVERDOGO TELA PO NEPODVIZHNOI POVERKHNOSTI)

PMM Vol. 29, No. 3, 1965, pp. 573–583

Iu. P. BYCHKOV
(Moscow)

(*Received July 10, 1964*)

The problem of the rolling of a rigid body with a finite surface, on another fixed surface, has been investigated by many authors. We should first note the investigations [1–7], which contain almost all the basic results obtained on this problem up to the present time. This paper, using the method of Woronetz, considers the case of the rolling of a body of revolution[1] on the surface of revolution and points out some new cases of integration.

1. Consider systems of rectangular coordinate axes $Ox_1x_2x_3$ and $O_1x_1^1x_2^1x_3^1$ ($\mathbf{i}_1$, $\mathbf{i}_2$, $\mathbf{i}_3$; and $\mathbf{i}_1^1$, $\mathbf{i}_2^1$, $\mathbf{i}_3^1$ are the respective unit vectors), rigidly connected to the rigid body and the base surface respectively (all coordinate systems in the paper are left-handed). This enables the position of the body with the coordinates x_{10}^1, x_{20}^1, x_{30}^1 of point O in the system $O_1x_1^1x_2^1x_3^1$ and Euler angles ϕ, ψ, θ (pure rotation, precession and nutation) between the above axes to be determined. The components of velocity vector $\mathbf{v}_0$ of point O and the body angular velocity vector ω on axes $Ox_1x_2x_3$ are denoted by k, l, m and p, q, r.

Further, considering for the points of surface S, bounding the rigid body, a radius-vector ρ starting from point O and Gauss coordinates q^1, q^2, gives its equation in the form

$$\varrho = \varrho\,(q^1, q^2) \qquad (\varrho = x_1\mathbf{i}_1 + x_2\mathbf{i}_2 + x_3\mathbf{i}_3) \tag{1.1}$$

and the coefficients of the first two quadratic forms will be denoted by a_{11}, a_{22}, b_{11}, b_{22} (for simplicity, we assume that the coordinate lines of the surface are lines of curvature). To the point of contact M on surface S we shall attach a moving datum Mq^1q^2n with unit vectors, directed along the tangent to the coordinates and the normal,

$$\mathbf{e}_1 = \frac{1}{\sqrt{a_{11}}}\varrho_1, \quad \mathbf{e}_2 = \frac{1}{\sqrt{a_{22}}}\varrho_2, \quad \mathbf{e}_3 = \frac{1}{\sqrt{a_{11}a_{22}}}(\varrho_1 \times \varrho_2) \quad \left(\varrho_\alpha = \frac{\partial}{\partial q^\alpha}\varrho\right) \tag{1.2}$$

---

[1] A body of revolution is understood here to mean a rigid body, bounded by the surface of revolution, the axis of which passes through the center of inertia and is also the dynamic axis of symmetry of the body.

and shall denote the components of vector ρ on these axes as

$$\xi = \frac{1}{\sqrt{a_{11}}}\rho\frac{\partial\rho}{\partial q^1}, \quad \eta = \frac{1}{\sqrt{a_{22}}}\rho\frac{\partial\rho}{\partial q^2}, \quad \varepsilon \quad (\rho^2 = x_1{}^2 + x_2{}^2 + x_3{}^2) \tag{1.3}$$

We shall also introduce the nine cosines of the angles between axes $Ox_1x_2x_3$ and Mq^1q^2n using the expression

$$\mathbf{i}_k = l_{1k}\mathbf{e}_1 + l_{2k}\mathbf{e}_2 + l_{3k}\mathbf{e}_3 \qquad (k = 1, 2, 3)$$

The above comments for surface S, bounding the rigid body, are also valid for the base surface S^1 (corresponding variables are denoted by the same letters, but indexed). Further, following Woronetz, we shall determine the position of the body using the generalized coordinates q^1, q^2, q_1^1, q_1^2, ϑ (the first four are the Gauss coordinates of point M, and ϑ is the angle between the axes q^1 and q_1^2 at the same point).

The projections of the body angular velocity ω on the moving datum axes Mq^1q^2n are given by the expressions[1] (here and below, the upper and lower signs correspond to the case $\mathbf{e}_3 = \mathbf{e}_3^1$ and $\mathbf{e}_3 = -\mathbf{e}_3^1$ respectively):

$$\sigma = -\frac{b_{22}}{\sqrt{a_{22}}}q^{2\cdot} \pm \frac{b_{22}^1}{\sqrt{a_{22}^1}}q_1^{2\cdot}\sin\vartheta - \frac{b_{11}^1}{\sqrt{a_{11}^1}}q_1^{1\cdot}\cos\vartheta$$

$$\tau = \frac{b_{11}}{\sqrt{a_{11}}}q^{1\cdot} - \frac{b_{11}^1}{\sqrt{a_{11}^1}}q_1^{1\cdot}\sin\vartheta \mp \frac{b_{22}^1}{\sqrt{a_{22}^1}}q_1^{2\cdot}\cos\vartheta \tag{1.4}$$

$$n = \frac{1}{2\sqrt{a_{11}a_{22}}}\left(\frac{\partial a_{11}}{\partial q^2}q^{1\cdot} - \frac{\partial a_{22}}{\partial q^1}q^{2\cdot}\right) \mp \frac{1}{2\sqrt{a_{11}^1a_{22}^1}}\left(\frac{\partial a_{11}^1}{\partial q_1^2}q_1^{1\cdot} - \frac{\partial a_{22}^1}{\partial q_1^1}q_1^{2\cdot}\right) - \frac{d\vartheta}{dt}$$

In this problem we have the relations (1.5)

$$\sqrt{a_{11}^1}q_1^{1\cdot} = \pm\sqrt{a_{11}}q^{1\cdot}\sin\vartheta \mp \sqrt{a_{22}}q^{2\cdot}\cos\vartheta, \quad \sqrt{a_{22}^1}q_1^{2\cdot} = \sqrt{a_{11}}q^{1\cdot}\cos\vartheta + \sqrt{a_{22}}q^{2\cdot}\sin\vartheta$$

and equations (1.4), using the above, can be written as

$$\sigma = -\Delta_{12}\sqrt{a_{11}}q^{1\cdot} - \Delta_{22}\sqrt{a_{22}}q^{2\cdot}, \qquad \tau = \Delta_{11}\sqrt{a_{11}}q^{1\cdot} + \Delta_{21}\sqrt{a_{22}}q^{2\cdot}$$

$$n = -\vartheta^{\cdot} + \Delta_1\sqrt{a_{11}}q^{1\cdot} - \Delta_2\sqrt{a_{22}}q^{2\cdot}$$

where

$$\Delta_{11} = \frac{b_{11}}{a_{11}} \mp \frac{b_{11}^1}{a_{11}^1}\sin^2\vartheta \mp \frac{b_{22}^1}{a_{22}^1}\cos^2\vartheta, \quad \Delta_{22} = \frac{b_{22}}{a_{22}} \mp \frac{b_{22}^1}{a_{22}^1}\sin^2\vartheta \mp \frac{b_{11}^1}{a_{11}^1}\cos^2\vartheta$$

$$\Delta_{12} = \Delta_{21} = \mp\left(\frac{b_{22}^1}{a_{22}^1} - \frac{b_{11}^1}{a_{11}^1}\right)\sin\vartheta\cos\vartheta$$

$$\Delta_1 = \frac{1}{2\sqrt{a_{22}}}\frac{\partial\log a_{11}}{\partial q^2} - \frac{\sin\vartheta}{2\sqrt{a_{22}^1}}\frac{\partial\log a_{11}^1}{\partial q_1^2} \pm \frac{\cos\vartheta}{2\sqrt{a_{11}^1}}\frac{\partial\log a_{22}^1}{\partial q_1^1}$$

$$\Delta_2 = \frac{1}{2\sqrt{a_{11}}}\frac{\partial\log a_{22}}{\partial q^1} \mp \frac{\sin\vartheta}{2\sqrt{a_{11}^1}}\frac{\partial\log a_{22}^1}{\partial q_1^1} - \frac{\cos\vartheta}{2\sqrt{a_{22}^1}}\frac{\partial\log a_{11}^1}{\partial q_1^2}$$

[1] The first terms in these expressions are the projections of the angular velocity of axes $Ox_1x_2x_3$ relative to Mq^1q^2n on the same axes Mq^1q^2n, they are denoted below by σ_1, τ_1 and n_1.

The equations of motion of a rigid body, of finite surface, moving on another fixed surface, when point O is the center of inertia, are

$$\frac{d}{dt}\frac{\partial\Theta}{\partial\sigma}+(\tau-\tau_1)\frac{\partial\Theta}{\partial n}-(n-n_1)\frac{\partial\Theta}{\partial\tau}+M(\xi\tau-\eta\sigma)\sqrt{a_{22}}\,q^{2\cdot}=P_1$$

$$\frac{d}{dt}\frac{\partial\Theta}{\partial\tau}+(n-n_1)\frac{\partial\Theta}{\partial\sigma}-(\sigma-\sigma_1)\frac{\partial\Theta}{\partial n}-M(\xi\tau-\eta\sigma)\sqrt{a_{11}}\,q^{1\cdot}=P_2 \qquad (1.6)$$

$$\frac{d}{dt}\frac{\partial\Theta}{\partial n}+(\sigma-\sigma_1)\frac{\partial\Theta}{\partial\tau}-(\tau-\tau_1)\frac{\partial\Theta}{\partial\sigma}+$$

$$+M\varepsilon\left(\sqrt{a_{11}}\,q^{1\cdot}\sigma+\sqrt{a_{22}}\,q^{2\cdot}\tau\right)-M\left(\rho\frac{\partial\rho}{\partial q^1}\,q^{1\cdot}+\rho\frac{\partial\rho}{\partial q^2}\,q^{2\cdot}\right)n=P_3$$

$$P_\alpha=\frac{\triangle_{\alpha2}}{\sqrt{a_{11}}\,R}\frac{\partial U}{\partial q^1}-\frac{\triangle_{\alpha1}}{\sqrt{a_{22}}\,R}\frac{\partial U}{\partial q^2}+\frac{1}{R}(\triangle_{\alpha2}\triangle_{.1}+\triangle_{\alpha1}\triangle_{.2})\frac{\partial U}{\partial\vartheta}\pm \qquad (\alpha=1,2)$$

$$\pm\frac{1}{\sqrt{a_{11}^1}\,R}(\triangle_{\alpha2}\sin\vartheta+\triangle_{\alpha1}\cos\vartheta)\frac{\partial U}{\partial q_1^1}+\frac{1}{\sqrt{a_{22}^1}\,R}(\triangle_{\alpha2}\cos\vartheta-\triangle_{\alpha1}\sin\vartheta)\frac{\partial U}{\partial q_1^2}$$

$$P_3=-\frac{\partial U}{\partial\vartheta} \qquad (1.7)$$

Here Θ is the kinetic energy, derived using (1.5), U is the forcing function and $R = \triangle_{11}\triangle_{22}-\triangle_{12}^2$.

Assuming that the axes $Ox_1x_2x_3$ are the principal central inertia axes and denoting by A, B, C the principal central moments of inertia, we get

$$2\Theta=M\rho^2(\sigma^2+\tau^2+n^2)-M(\xi\sigma+\eta\tau+\varepsilon n)^2+$$

$$+A(\sigma l_{11}+\tau l_{21}+nl_{31})^2+B(\sigma l_{12}+\tau l_{22}+nl_{32})^2+C(\sigma l_{13}+\tau l_{23}+nl_{33})^2 \qquad (1.8)$$

Also, the cosiness of the angles of the axis Ox_1 with axes $O_1x_1^1$, $O_1x_2^1$, and $O_1x_3^1$ are (the corresponding cosiness for axes Ox_2 and Ox_3 are obtained by changing l_{11}, l_{21}, l_{31} to l_{12}, l_{22}, l_{32} and l_{13}, l_{23}, l_{33}) (1.9)

$$l_{31}(\pm l_{3k}^1)+l_{21}(\mp l_{1k}^1\cos\vartheta+l_{2k}^1\sin\vartheta)+l_{11}(\pm l_{1k}^1\sin\vartheta+l_{2k}^1\cos\vartheta) \quad (k=1,2,3)$$

The expressions of the coordinates of the center of inertia in the system $O_1x_1^1x_2^1x_3^1$ are

$$x_{k0}^1=x_k^1-[(\pm l_{1k}^1\sin\vartheta+l_{2k}^1\cos\vartheta)\xi+$$

$$+(\mp l_{1k}^1\cos\vartheta+l_{2k}^1\sin\vartheta)\eta+(\pm l_{3k}^1)\varepsilon] \quad (k=1,2,3) \qquad (1.10)$$

2. We shall now consider some problems of the rolling of a body of revolution bounded by the surface of revolution

$$x_1=u\cos v, \quad x_2=u\sin v, \quad x_3=f(u)$$

on another surface of revolution

$$x_1^1=u_1\cos v_1, \quad x_2^1=u_1\sin v_1, \quad x_3^1=f^1(u_1)$$

in which the forcing function is of the form $U(u,\vartheta,u_1)$. Such a case occurs if the body is under the action of gravity and the axis $O_1x_3^1$ is vertical, and also if the body is acted on by forces, the resultant of which is directed from the center of inertia to the point O_1 of the axis of symmetry of the base, and depends only on the distance between these points.

We shall first consider the problem for a body bounded by a sphere. The equation of the sphere, in the system $Ox_1x_2x_3$ has the form (l is the coordinate of the geometric center relative to axis Ox_3)

$$x_1 = R \sin u \cos v, \quad x_2 = R \sin u \sin v, \quad x_3 - l = R \cos u \tag{2.1}$$

Using these equations we get

$$a_{11} = R^2, \quad a_{22} = R^2 \sin^2 u, \quad b_{11} = -R, \quad b_{22} = -R \sin^2 u$$
$$l_{13} = -\sin u, \quad l_{23} = 0, \quad l_{33} = \cos u, \quad \rho^2 = R^2 + 2lR \cos u + l^2 \tag{2.2}$$
$$\xi = -l \sin u, \quad \eta = 0, \quad \varepsilon = R + l \cos u \tag{2.3}$$

With equations (2.2) and (2.3), the expression for the kinetic energy can be written as

$$\begin{aligned} 2\Theta = {} & [A + M(R^2 + l^2 + 2Rl \cos u)]\,\tau^2 + [M(R + l\cos u)^2 + C \sin^2 u + \\ & + A \cos^2 u]\,\sigma^2 + 2\,[(A - C)\cos u + Ml(R + l \cos u)] \sin u\sigma n + \\ & + [C \cos^2 u + (A + Ml^2)\sin^2 u]\, n^2 \end{aligned} \tag{2.4}$$

We shall now derive the equations of motion (1.6) of the body of revolution, bounded by a sphere, on any fixed surface.[1]

$$\begin{aligned} & \frac{d}{dt}\frac{\partial\Theta}{\partial\sigma} + (\tau + u^\cdot)\frac{\partial\Theta}{\partial n} - (n + v^\cdot \cos u)\frac{\partial\Theta}{\partial\tau} - MlR \sin^2 u\tau v^\cdot = P_1 \\ & \frac{d}{dt}\frac{\partial\Theta}{\partial\tau} + (n + v^\cdot \cos u)\frac{\partial\Theta}{\partial\sigma} - (\sigma - v^\cdot \sin u)\frac{\partial\Theta}{\partial n} + MlR \sin u\tau u^\cdot = P_2 \\ & \frac{d}{dt}\frac{\partial\Theta}{\partial n} + (\sigma - v^\cdot \sin u)\frac{\partial\Theta}{\partial\tau} - (\tau + u^\cdot)\frac{\partial\Theta}{\partial\sigma} + M(R + l\cos u)(Ru^\cdot\sigma + R \sin u v^\cdot \tau) + \\ & \qquad + MlR \sin u n u^\cdot = P_3 \end{aligned} \tag{2.5}$$

We shall assume that the base surface is also a sphere, the equations of which in the system $O_1x_1^1x_2^1x_3^1$ are

$$x_1^1 = R_1 \sin u_1 \cos v_1, \quad x_2^1 = R_1 \sin u_1 \sin v_1, \quad x_3^1 = R_1 \cos u_1 \tag{2.6}$$

and for which, based on the above calculations (2.2), we have the analogous expressions

$$\begin{aligned} & a_{11}^1 = R_1^2, \quad a_{22}^1 = R_1^2 \sin^2 u_1, \quad b_{11}^1 = -R_1, \quad b_{22}^1 = -R_1 \sin^2 u_1 \\ & l_{13}^1 = -\sin u_1, \quad l_{23}^1 = 0, \quad l_{33}^1 = \cos u_1 \end{aligned} \tag{2.7}$$

If, in the rolling body, the concave (convex) side of the bounding sphere is in contact with the convex (concave) side of the base-sphere, we have $e_3 = e_3^1$ (this case is presented below); however, if the surfaces are in contact on their concave sides we have $e_3 = -e_3^1$.

The equations of relations (1.5) in this problem are

$$R_1 u_1^\cdot = Ru^\cdot \sin\vartheta - R \sin u v^\cdot \cos\vartheta, \qquad R_1 \sin u_1 v_1^\cdot = Ru^\cdot \cos\vartheta + R\sin u v^\cdot \sin\vartheta$$

The projections of the body angular velocity on the axes of the moving datum (1.4) become here

---

[1] If in the rolling body there is a rotor turning about axis Ox_3 with constant velocity ω', the center of inertia of which coincides with the center of inertia of the body, then the following terms are added to the right sides of equations (1.6) (M, A, B, C are the mass and moments of inertia of the whole system, C' is the moment of inertia of the rotor about axis Ox_3 and Ox_3, $\varkappa = C'\omega'$):

$$\varkappa\,(nl_{23} - \tau l_{33}),\ \varkappa\,(\sigma l_{33} - nl_{13}), \qquad \varkappa\,(\tau l_{13} - \sigma l_{23})$$

$$\sigma = -\frac{R - R_1}{R_1} \sin u v^{\cdot}, \quad \tau = \frac{R - R_1}{R_1} u^{\cdot}, \quad n = -\cos u v^{\cdot} + \cos u_1 v_1^{\cdot} - \vartheta^{\cdot} \quad (2.9)$$

Finally, the forcing function of gravity (to be specific, we shall take axis $O_1 x_3^1$ vertically downwards), according to (1.10), (2.3) and (2.7), is here

$$U = Mg\,[(R_1 - R - l\cos u)\cos u_1 - l \sin u \sin u_1 \sin \vartheta] \quad (2.10)$$

and the forcing function of the "central forces" with centers O and O_1, as shown in [2], is a function of the single coordinate u.

Let us next consider the question of integrating the system of equations (2.5), (2.8) and (2.9) when the forcing function of the given forces depends only on one coordinate u ("central forces"). This problem can be investigated in two stages: first by considering the magnitude of u, v, σ, τ, n and then considering the question of determining the remaining variables u_1, v_1 and ϑ. Assume that the center of inertia is at the geometric center of the sphere, bounding the body. In this case the problem of calculating u, v, σ, τ and n leads to the integration of two equations

$$\frac{1}{2}(A - C)\sin 2u \frac{dn}{du} + (MR^2 + A\cos^2 u + C\sin^2 u)\frac{d\sigma}{du} +$$
$$+ \left\{[MR^2 + (A - C)\cos^2 u]\left(1 - \frac{R}{R_1}\right) + A\cos^2 u + C\sin^2 u\right\} n +$$
$$+ \left[\frac{1}{2}(A - C)\left(\frac{R}{R_1} - 2\right)\sin 2u + (A + MR^2)\cot u\right]\sigma = \varkappa\left(1 - \frac{R}{R_1}\right)\cos u \quad (2.11)$$

$$(A\sin^2 u + C\cos^2 u)\frac{dn}{du} + \frac{1}{2}(A - C)\sin 2u \frac{d\sigma}{du} -$$
$$- \left\{\frac{1}{2}\sin 2u\left(\frac{R}{R_1} - 2\right)n - \left[\sin^2 u\left(\frac{R}{R_1} - 1\right) + \cos^2 u\right]\sigma\right\}(A - C) = \varkappa\left(1 - \frac{R}{R_1}\right)\sin u$$

which are obtained from the first and third equation of the system (2.5) (divided by $u^{\cdot}$ and rearranged).

Adding the first equation (2.11), multiplied by $-\tan u$, to the second we get

$$-(MR^2 + C)\left(\tan u \frac{d\sigma}{du} + \sigma\right) + C\frac{dn}{du} + \left[\left(\frac{R}{R_1} - 1\right)MR^2 - C\right]\tan u\, n = 0$$

and adding the second equation, multiplied by $\tan u$, to the first

$$(MR^2 + A)\frac{d\sigma}{du} + \left[\left(\frac{R}{R_1} - 1\right)(A - C)\tan u + (MR^2 + A)\cot u\right]\sigma +$$
$$+ A\tan u \frac{dn}{du} + \left[\left(\frac{R}{R_1} - 1\right)(C - A - MR^2) + A\right] n = \left(1 - \frac{R}{R_1}\right)\frac{\varkappa}{\cos u}$$

By changing the variables to

$$\xi = \sin u, \quad \eta = \sin u\sigma, \quad \zeta = \cos u n \quad (2.12)$$

the above equations become

$$-(MR^2 + C)\frac{d\eta}{d\xi} + C\frac{d\zeta}{d\xi} + \left(\frac{R}{R_1} - 1\right)MR^2\frac{\xi}{1 - \xi^2}\zeta = 0$$

$$(MR^2 + A)\frac{d\eta}{d\xi} + \left(\frac{R}{R_1} - 1\right)(A - C)\frac{\xi}{1 - \xi^2}\eta + A\frac{d\zeta}{d\xi} +$$
$$+ \left(1 - \frac{R}{R_1}\right)(MR^2 + A - C)\frac{\xi}{1 - \xi^2}\zeta + A\frac{1}{1 - \xi^2}\left(\frac{d\zeta}{d\xi} + \frac{\xi}{1 - \xi^2}\zeta\right) = \varkappa\left(1 - \frac{R}{R_1}\right)\frac{\xi}{1 - \xi^2}$$

Making one more substitution $x = 1 - \xi^2$, we obtain from the first equation

$$\frac{d\eta}{dx} = \frac{C}{MR^2 + C}\frac{d\zeta}{dx} - \frac{1}{2}\left(\frac{R}{R_1} - 1\right)\frac{MR^2}{MR^2 + C}\frac{\zeta}{x} \tag{2.13}$$

and from the second

$$(MR^2 + A)\, x\frac{d\eta}{dx} - \frac{1}{2}\left(\frac{R}{R_1} - 1\right)(A - C)\,\eta - Ax\frac{d\zeta}{dx} + \frac{1}{2}\left(\frac{R}{R_1} - 1\right)(MR^2 + A - C)\,\zeta +$$
$$+ A\left(\frac{d\zeta}{dx} - \frac{1}{2}\frac{\zeta}{x}\right) = \frac{1}{2}\varkappa\left(\frac{R}{R_1} - 1\right)$$

Differentiating it with respect to x, and substituting equation (2.13), we obtain a Fuch's equation [8,9] with three singular points $a' = 0$, $b' = -m$, and $c' = \infty$ $(A \neq C)$

$$x^2(m + x)\frac{d^2\zeta}{dx^2} - x\left(\frac{1}{2}m - x\right)\frac{d\zeta}{dx} + \left(\frac{1}{2}m - k^2x\right)\zeta = 0$$
$$\left(m = \frac{A(MR^2 + C)}{MR^2(C - A)},\ k = \frac{1}{2}\left(\frac{R}{R_1} - 1\right)\right)$$

Equating this to the general equation of this type in Papperits' form [8, 9]

$$\frac{d^2\zeta}{dx^2} + \left(\frac{1 - \alpha - \alpha'}{x - a'} + \frac{1 - \beta - \beta'}{x - b'}\right)\frac{d\zeta}{dx} +$$
$$+ \left[\frac{(a' - b')\,\alpha\alpha'}{x - a'} + \frac{(b' - a')\,\beta\beta'}{x - b'} + \gamma\gamma'\right]\frac{\zeta}{(x - a')(x - b')} = 0$$

we find

$$\alpha = 1,\quad \beta = 0,\quad \gamma = k,\quad \alpha' = 1/2,\quad \beta' = -1/2,\quad \gamma' = -k$$

This enables us to write the required solution in the form

$$P\begin{Bmatrix} 0 & -m & \infty & \\ 1 & 0 & k & x \\ 1/2 & -1/2 & -k & \end{Bmatrix} = xP\begin{Bmatrix} 0 & 1 & \infty & \\ 0 & 0 & 1 + k & (-x/m) \\ -1/2 & -1/2 & 1 - k & \end{Bmatrix}$$

It follows that the function ζ has the form

$$\zeta = \cos^2 u\left\{C_1F\left[1 + \frac{1}{2}\left(\frac{R}{R_1} - 1\right),\ 1 - \frac{1}{2}\left(\frac{R}{R_1} - 1\right);\ \frac{3}{2};\ -\frac{\cos^2 u}{m}\right] +\right.$$
$$\left.+ C_2\frac{1}{\cos u}F\left[\frac{1}{2} + \frac{1}{2}\left(\frac{R}{R_1} - 1\right),\ \frac{1}{2} - \frac{1}{2}\left(\frac{R}{R_1} - 1\right);\ \frac{1}{2};\ -\frac{\cos^2 u}{m}\right]\right\}$$

and for the case $R_1 = \infty$ (the base-sphere becomes a plane)

$$\zeta = \cos u\left[C_1\cos u\left(1 + \frac{\cos^2 u}{m}\right)^{-1/2} + C_2\right]$$

Now, considering the known relation

$$\frac{d}{dy}F(a, b; c; y) = \frac{ab}{c}F(a + 1,\ b + 1;\ c + 1;\ y)$$

and using equation (2.13) we can determine the quantities η and σ, as functions of the variable u. Afterwards, substituting the expressions for n and σ into the integral of kinetic energies, the problem of determining the variable u, as a function of time, is reduced to a quadrature. This completes the first stage of integrating the system (2.5), (2.8) and (2.9).

3. We shall now investigate problems, similar to the ones studied by Noether [4], which are also of the above type. Namely, we shall assume that the moving body is a "uniform sphere". In this case equations (’.2) are unchanged, equations (2.3) become

$$\rho^2 = R^2, \quad \xi = 0, \quad \eta = 0, \quad \varepsilon = R \tag{3.1}$$

and the expression of the kinetic energy

$$2\Theta = (A + MR^2)(\sigma^2 + \tau^2) + An^2 \tag{3.2}$$

Using (1.4), (2.2) and (3.1), based on equation (1.6) we obtain the equation of motion of a "uniform sphere" on any fixed surface as

$$\begin{gathered}(MR^2 + A)\,\sigma^\cdot + (\tau + u^\cdot)\,An - (n + v^\cdot \cos u)\,(MR^2 + A)\,\tau = P_1\\ (MR^2 + A)\,\tau^\cdot + (n + v^\cdot \cos u)\,(MR^2 + A)\,\sigma - (\sigma - v^\cdot \sin u)\,An = P_2\\ n^\cdot - (u^\cdot \sigma + \sin u v^\cdot \tau) = P_3\end{gathered} \tag{3.3}$$

Consider the problem of the motion of this body under gravity on the surface of paraboloid of revolution [1]; the method of solution of an analagous problem for the case of any other surface of revolution will be the same as in the given particular case.

The base-paraboloid is described in the system $O_1x_1^1x_2^1x_3^1$ by the equations

$$x_1^1 = u_1 \cos v_1, \quad x_2^1 = u_1 \sin v_1, \quad x_3^1 = -\frac{1}{2p}u_1^2 \tag{3.4}$$

The following are then derived

$$a_{11}^1 = \frac{p^2 + u_1^2}{p^2}, \quad a_{22}^1 = u_1^2, \quad b_{11}^1 = -\frac{1}{\sqrt{p^2 + u_1^2}}, \quad b_{22}^1 = -\frac{u_1^2}{\sqrt{p^2 + u_1^2}}$$

$$l_{13}^1 = -\frac{u_1}{\sqrt{p^2 + u_1^2}}, \quad l_{23}^1 = 0, \quad l_{33}^1 = \frac{p}{\sqrt{p^2 + u_1^2}} \tag{3.5}$$

If the concave (convex) side of the bounding sphere of the body is in contact with the convex (concave) side of the paraboloid, we have $e_3 = e_3^1$ (this case is presented below). The equations of constraints are here

$$\frac{\sqrt{p^2 + u_1^2}}{p}\,u_1^\cdot = Ru^\cdot \sin\vartheta - R\sin u v^\cdot \cos\vartheta, \qquad u_1 v_1^\cdot = Ru^\cdot \cos\vartheta + R\sin u v^\cdot \sin\vartheta \tag{3.6}$$

The projections of the body angular velocity on the axes of the moving datum are

$$\begin{gathered}\sigma = -\frac{Ru_1^2}{\sqrt{p^2 + u_1^2}\,(p^2 + u_1^2)}\sin\vartheta\cos\vartheta u^\cdot - \\ -\left[-\sin u + \frac{R\sin u}{\sqrt{p^2 + u_1^2}\,(p^2 + u_1^2)}(p^2 + u_1^2\sin^2\vartheta)\right]v^\cdot\\ \tau = \left[-1 + \frac{R}{\sqrt{p^2 + u_1^2}\,(p^2 + u_1^2)}(p^2 + u_1^2\cos^2\vartheta)\right]u^\cdot + \frac{R\sin u\,u_1^2}{\sqrt{p^2 + u_1^2}\,(p^2 + u_1^2)}\sin\vartheta\cos\vartheta v^\cdot\\ n = -\cos u v^\cdot + \frac{p}{\sqrt{p^2 + u_1^2}}\,v_1^\cdot - \vartheta^\cdot\end{gathered} \tag{3.7}$$

The forcing function of gravity [2], according to (3.1) and (3.5) can be written as

---

[1] In [4], the center of gravity and not the point of contact moves on a paraboloid of revolution.

[2] We investigate the case of the body motion on the concave side of the paraboloid, with the axis $O_1x_3^1$ directed vertically downwards.

$$U = Mgx_{30}^1 = -Mg\left(\frac{1}{2p}u_1^2 + \frac{Rp}{\sqrt{p^2+u_1^2}}\right) \tag{3.8}$$

Using (1.7) we find

$$P_1 = f(u_1)\cos\vartheta, \qquad P_2 = f(u_1)\sin\vartheta, \quad P_3 = 0$$

We shall now proceed to integrate the system of equations (3.3), (3.6) and (3.7) directly. First note the relations obtained from (3.6) and (3.7)

$$\tau\cos\vartheta - \sigma\sin\vartheta = u_1\left(\frac{1}{\sqrt{p^2+u_1^2}} - \frac{1}{R}\right)v_1^{\cdot}$$

$$\tau\sin\vartheta + \sigma\cos\vartheta = \frac{\sqrt{p^2+u_1^2}}{p}\left(\frac{p^2}{\sqrt{p^2+u_1^2}\,(p^2+u_1^2)} - \frac{1}{R}\right)u_1^{\cdot}$$

Next add the first equation (3.3), multiplied by $-\sin\vartheta$, to the second equation (3.3), multiplied by $\cos\vartheta$, and transform the sum using (3.6), (3.7) and the above relations. The resulting equation is ($A = Mk^2$)

$$u_1\left(-\frac{1}{R} + \frac{1}{\sqrt{p^2+u_1^2}}\right)\frac{p^2+u_1^2}{p}\frac{dv_1^{\cdot}}{du_1} +$$

$$+2\left(-\frac{1}{R} + \frac{p^2}{\sqrt{p^2+u_1^2}\,(p^2+u_1^2)}\right)\frac{p^2+u_1^2}{p}v_1^{\cdot} - \frac{k^2}{R^2+k^2}n = 0$$

Using the same (3.6) and (3.7), the third equation (3.3) becomes

$$\frac{dn}{du_1} = -\frac{u_1^3}{pR\,(p^2+u_1^2)}v_1^{\cdot}$$

Making the substitution $x = p\,(p^2+u_1^2)^{-1/2}$, the second equation gives

$$v_1^{\cdot} = \frac{R}{p}\frac{x^3}{1-x^2}\frac{dn}{dx}, \qquad \frac{dv_1^{\cdot}}{dx} = \frac{R}{p}\left[\frac{x^3}{1-x^2}\frac{d^2n}{dx^2} + \frac{x^2(3-x^2)}{(1-x^2)^2}\frac{dn}{dx}\right] \tag{3.9}$$

and the first equation is written as

$$\frac{1-x^2}{x}\left(1-\frac{R}{p}x\right)\frac{p}{R}\frac{dv_1^{\cdot}}{dx} - \frac{2}{x^2}\left(1-\frac{R}{p}x^3\right)\frac{p}{R}v_1^{\cdot} - \frac{k^2}{R^2+k^2}n = 0$$

Eliminating the quantities $v_1^{\cdot}$ and $dv_1^{\cdot}/dx$, from the last equation using relations (3.0), we obtain, for the solution of function n, the following Fuch's equation [8, 9]:

$$\left(x-\frac{p}{R}\right)x^2\frac{d^2n}{dx^2} + \left(3x-\frac{p}{R}\right)x\frac{dn}{dx} + \frac{p}{R}a^2n = 0 \qquad \left(a^2 = \frac{k^2}{R^2+k^2}\right)$$

Its solution has the form

$$n = x^a\,[C_1F(2+a, a; 1+2a; y) + C_2y^{-2a}F(2-a, -a; 1-2a; y)] \qquad (y = Rp^{-1}x)$$

The second unknown function $v_1^{\cdot}$ is now obtained from the first relation (3.9).

Note further that for the motion of a body, bounded by a sphere, on any surface, we can derive the following equation using (1.4) and (1.5)

$$\sigma^2 + \tau^2 = u_1^{\cdot 2}\left(\frac{a_{11}^1}{R^2} + \frac{(b_{11}^1)^2}{a_{11}^1} \pm 2\frac{b_{11}^1}{R}\right) + v_1^{\cdot 2}\left(\frac{a_{22}^1}{R^2} + \frac{(b_{22}^1)^2}{a_{22}^1} \pm 2\frac{b_{22}^1}{R}\right) \tag{3.10}$$

Next, substitute this equation and the functions n and $v_1^{\cdot}$ obtained above into the integral of kinetic energies. We thus find that the problem of determining the variable u_1 (as a function of time) is reduced to a quadrature.

It is still left to find the variables u, v and ϑ. But from relations (1.5) and equations (1.4) for the motion of a body, bounded by a sphere, on any surface, we obtain three equations

$$Ru^{\cdot} = \pm\sqrt{a_{11}^1}u_1^{\cdot}\sin\vartheta + \sqrt{a_{22}^1}v_1^{\cdot}\cos\vartheta$$

$$R\sin u v^{\cdot} = \mp\sqrt{a_{11}^1}\,u_1^{\cdot}\cos\vartheta + \sqrt{a_{22}^1}\,v_1^{\cdot}\sin\vartheta \tag{3.11}$$

$$\vartheta^{\cdot} = -n \mp \frac{1}{2\sqrt{a_{11}^1 a_{22}^1}}\left(\frac{\partial a_{11}^1}{\partial v_1}u_1^{\cdot} - \frac{\partial a_{22}^1}{\partial u_1}v_1^{\cdot}\right) - v^{\cdot}\cos u$$

i.e., we have here the same situation as was studied by Woronetz. This problem, incidentally, has much in common with the previous one, in particular here the coordinates u_1, v_1 and u, v and ϑ are obtained in the same way as we obtained the coordinates u, v and u_1, v_1 and ϑ (respectively) in the second section.

We shall now present another example of the same type (a simple, but nevertheless interesting one), the problem of the motion of a heavy "uniform sphere" on a fixed sphere.

Proceeding here in a similar fashion to the paraboloid problem, using (3.3), (2.8) and (2.9)[1] we obtain the equations

$$\pm(MR^2 + A)\frac{R \mp R_1}{R}\left[\frac{d}{dt}(v_1^{\cdot}\sin u_1) + u_1^{\cdot}v_1^{\cdot}\cos u_1\right] - Anu_1^{\cdot} = 0 \qquad n = \text{const} \tag{3.12}$$

Multiplying the first by $\sin u_1$ and integrating the product we get ($A = Mk^2$, $\varkappa$ is a constant)

$$v_1^{\cdot}\sin^2 u_1 = \beta - b\cos u_1$$

$$\left(\beta = \frac{\varkappa R}{M(R^2 + k^2)(R \mp R_1)}, \quad b = \pm\frac{k^2Rn}{(R^2 + k^2)(R \mp R_1)}\right) \tag{3.13}$$

Next, using (2.8) – (2.10) and (3.2), the integral of kinetic energy can be written as

$$u_1^{\cdot 2} + v_1^{\cdot 2}\sin^2 u_1 = \alpha - a\cos u_1$$

$$\left(\alpha = \frac{2h - An^2}{M}\frac{R^2}{(R^2 + k^2)(R \mp R_1)^2}, \quad a = \pm\frac{2R^2g}{(R^2 + k^2)(R \mp R_1)}\right) \tag{3.14}$$

and eliminating $v_1^{\cdot}$, by means of (3.13), and substituting $x = \cos u_1$, we obtain

$$\left(\frac{dx}{dt}\right)^2 = (\alpha - ax)(1 - x^2) - (\beta - bx)^2$$

The right-hand side third order polynomial is positive when $x = -\infty$, negative when $x = \pm 1$ and positive for some values of x, between -1 and $+1$, since in the actual motion u_1 has real values, i.e., it has roots

$$-\infty < e_3 < -1 < e_2 < e_1 < 1$$

Assuming, as usual, $x = e_1 + (e_2 - e_1)\omega^2$, we get the equation under investigation into the form

[1] For $e_3 = -e_3^1$ different equations are implied.

$$\pm \frac{d\omega}{\sqrt{(1-\omega^2)(1-k^2\omega^2)}} = \frac{1}{2}\sqrt{a(e_3-e_1)}\,dt \qquad \left(k^2 = \frac{e_2-e_1}{e_3-e_1}\right)$$

Thus, the problem of finding the variable x has been reduced to the inversion of an elliptic integral, and on the basis of this equation we can write

$$x = e_1 + (e_2 - e_1)\,\mathrm{sn}^2\,(1/_2\sqrt{a(e_3-e_1)}\,t)$$

The variable v_1 is now determined from relation (3.13).

Note that the integrals (3.12) – (3.14), which lead to the equation of this problem, have the same form as the classic integrals of the problem of the rotation of a rigid body about a fixed point in the case of Lagrange [10] (p. 176), and for $R_1 = 0$ (the problem degenerates into the above mentioned Lagrange case) from geometrical considerations and equations (3.11) – (3.14) we find ($e_3 = -e_3^1$): $u = \mathrm{const}$, $v = \mathrm{const}$, $n = -r$, $u_1 = \theta$, $v_1 = \psi$, $\vartheta = \varphi$.

Using equations (3.13) and (3.14), it is easy to indicate the shape of the curve described by the contact point on the fixed sphere ([10], p.178) between the parallels $x = e_1$ and $x = e_2$. We can pursue the analogy between these problems deeper (the problem of body motion in the Lagrange case and in the case of the rolling "uniform sphere"). In particular, if in the second problem with $t = 0$ we have $u_{10}^{\cdot} = 0$, $v_{10}^{\cdot} = 0$, $n_0 \neq 0$, and $u_{10} \neq 0$, we get the well-known particular solution, the detailed description of which presents no complications ([10], p. 181).

Next, we shall consider the question of stability, for the case of surfaces in contact on their convex sides, of the particular solution

$$u_1^{\cdot} = 0, \quad v_1^{\cdot} = 0, \quad n = n_0, \quad \sin u_1 = 0, \quad \cos u_1 = -1$$

The stability will be investigated with reference to the variables

$$u_1^{\cdot}, \quad \sin u_1 v_1^{\cdot}, \quad n, \sin u_1, \quad \cos u_1$$

assuming in the disturbed motion

$$u_1^{\cdot} = \xi, \quad \sin u_1 v_1^{\cdot} = \eta, \quad n = n_0 + \zeta, \quad \sin u_1 = \beta, \quad \cos u_1 = -1 + \delta$$

Note first that in this problem we have the first integrals

$$(MR^2 + A)\left(\frac{R+R_1}{R}\right)^2 (u_1^{\cdot 2} + \sin^2 u_1 v_1^{\cdot 2}) + An^2 - 2Mg\,(R+R_1)\cos u_1 = 2h$$

$$(MR^2 + A)\,\frac{R+R_1}{R}\,(\sin u_1 v_1^{\cdot})\sin u_1 - An\cos u_1 = \varkappa$$

$$\sin^2 u_1 + \cos^2 u_1 = 1, \qquad n = n_0$$

From the above it is easy to obtain the first integrals V_1, V_2, V_3 and V_4 also for the equations of the disturbed motion.

Now, to determine the sufficient conditions for stability we construct, by the method of Chetaev, Liapunov's function in the form of integral relations [11]

$$V = V_1 + 2\lambda V_2 - [Mg\,(R+R_1) + An_0\lambda]\,V_3 + \mu V_4^2 -$$

$$-2\,(An_0 + A\lambda)\,V_4 = (MR^2 + A)\left(\frac{R+R_1}{R}\right)^2 \xi^2 + (MR^2 + A)\left(\frac{R+R_1}{R}\right)^2 \eta^2 +$$

$$+ 2\lambda (MR^2 + A) \frac{R + R_1}{R} \eta\beta - [Mg (R + R_1) + An_0\lambda] \beta^2 + (A+\mu) \zeta^2 -$$

$$-2\lambda A\delta\zeta - [Mg (R + R_1) + An_0\lambda] \delta^2 \qquad (A^2 = (MR^2 + A) (A + \mu))$$

The function will be positive-definite if

$$A^2 n_0^2 - 4 (MR^2 + A) Mg (R + R_1) > 0$$

which for $R_1 = 0$ becomes Maievski's condition.

In conclusion, we note that for the case of "uniform sphere" motion on a fixed sphere under the action of a "central force" with centers O_1 and O, the variables u, v, σ, τ and n are obtained in the same manner as in section 2, and variables u_1 and v_1 as in section 3, i.e., the problem is fully solved by quadratures.

4. Next we shall investigate the problem of the motion of a body of revolution on a sphere in the case when the body rests on the sphere with its plane-bounded end.

Assuming first that the base-surface is any convex surface, we shall derive the equations of motion of this body. The equation of the surface in the system $Ox_1x_2x_3$ has the form (the coordinate of the point of intersection of the axis of symmetry of the body Ox_3 with the plane relative to the same axis will be denoted by d)

$$x_1 = u \cos v, \quad x_2 = u \sin v, \quad x_3 = d \tag{4.1}$$

We therefore find

$$\begin{gathered} a_{11} = 1, \quad a_{22} = u^2, \quad b_{11} = 0, \quad b_{22} = 0 \\ l_{13} = 0, \quad l_{23} = 0, \quad l_{33} = 1 \\ \rho^2 = u^2 + d^2, \quad \xi = u, \quad \eta = 0, \quad \varepsilon = d \end{gathered} \tag{4.2}$$

The expression for the kinetic energy is

$$2\Theta = (A + Md^2) \sigma^2 + (A + Mu^2 + Md^2) \tau^2 + (C + Mu^2) n^2 - 2Mdu\sigma n \tag{4.3}$$

Now, using (1.4) and (4.2), we derive the required equations of motion (a rotor is added to the body)

$$\begin{gathered} \frac{d}{dt}\frac{\partial\Theta}{\partial\sigma} + \tau\frac{\partial\Theta}{\partial n} - (n + v^{\cdot})\frac{\partial\Theta}{\partial\tau} + Mu^2\tau v^{\cdot} = P_1 - \varkappa\tau \\ \frac{d}{dt}\frac{\partial\Theta}{\partial\tau} + (n + v^{\cdot})\frac{\partial\Theta}{\partial\sigma} - \sigma\frac{\partial\Theta}{\partial n} - Mu\tau u^{\cdot} = P_2 + \varkappa\sigma \\ \frac{d}{dt}\frac{\partial\Theta}{\partial n} + \sigma\frac{\partial\Theta}{\partial\tau} - \tau\frac{\partial\Theta}{\partial\sigma} + Md (u^{\cdot}\sigma + uv^{\cdot}\tau) - Munu^{\cdot} = P_3 \end{gathered} \tag{4.4}$$

Next we shall return to the original problem, i.e., assume that the base-surface is a sphere, which is given by equations (2.6) in the system $O_1x_1^1x_2^1x_3^1$ and for which equations (2.7) hold. The axis Ox_3 is here taken in the direction of the sphere, therefore we have $e_3 = -e_3^1$. We shall now obtain the necessary kinematic relations. The relations (1.5) are here

$$R_1u_1^{\cdot} = -u^{\cdot}\sin\vartheta + uv^{\cdot}\cos\vartheta, \qquad R_1 \sin u_1 v_1^{\cdot} = u^{\cdot}\cos\vartheta + uv^{\cdot}\sin\vartheta \tag{4.5}$$

The projections of the body angular velocity on the axes of the moving datum are

$$\sigma = \frac{u}{R_1} v^{\cdot}, \quad \tau = -\frac{1}{R_1} u^{\cdot}, \quad n = -v^{\cdot} - \cos u_1 v_1^{\cdot} - \vartheta^{\cdot} \tag{4.6}$$

Let us also consider some details. First, we shall obtain the expressions for the projections of the body angular velocity and the projections of the velocity of the center of inertia of the body on axes $Ox_1x_2x_3$

$$p = \sigma \cos v - \tau \sin v, \quad q = \sigma \sin v + \tau \cos v, \quad r = n$$
$$k = u \sin vn - d\,(\sigma \sin v + \tau \cos v)$$
$$l = d\,(\sigma \cos v - \tau \sin v) - u \cos vn, \quad m = u\tau$$

Second, describing expressions (1.9) by means of (2.6) and (4.1) and comparing them with the corresponding expressions on p. 45 [12], we find the values of Euler's angles between axes $Ox_1x_2x_3$ and $O_1x_1^1x_2^1x_3^1$

$$\theta = \Pi - u_1, \quad \psi = -{}^1/_2\Pi + v_1, \quad \varphi = \Pi - v - \vartheta \tag{4.7}$$

Finally, we shall indicate the form of the forcing functions in some interesting cases. The gravity forcing function (the axis $O_1x_3^1$ is taken vertically upwards), according to (1.10), (4.2) and (2.7) is here

$$U = -Mg\,[(R_1 + d)\cos u_1 - u \sin u_1 \sin \vartheta] \tag{4.8}$$

and thus, using (1.7), we find

$$P_1 = Mgd \sin u_1 \cos \vartheta, \quad P_3 = -Mgu \sin u_1 \cos \vartheta$$
$$P_2 = Mgd \sin u_1 \sin \vartheta + Mgu \cos u_1$$

Also, the forcing function of "central forces" with centers O_1 and Q, as shown in [2], is a function of only one coordinate u.

We shall now proceed directly to the integration of the system of equations (4.4), (4.5) and (4.6) with the condition that the forcing function of the given forces depends only on one coordinate u, and that $d = 0$. The task of determining the quantities u, v, σ, τ and n reduces here to the integration of two first-order linear equations with two functions σ and n independent of u $(C = Mk^2)$

$$\frac{d\sigma}{du} - \frac{C-A}{A}\frac{n}{R_1} + \frac{\sigma}{u} = \frac{\varkappa}{A}\frac{1}{R_1}, \qquad (k^2 + u^2)\frac{dn}{du} + un - u^2\frac{\sigma}{R_1} = 0$$

which are obtained from the first and third equations (4.4) (they were divided by u).

Changing the variables to

$$x = \sqrt{k^2 + u^2}\left(\frac{C-A}{A}\right)^{1/2}, \quad y = u\sigma, \quad z = n\sqrt{k^2 + u^2}\left(\frac{C-A}{A}\right)^{1/2} \tag{4.9}$$

these equations become

$$\frac{dy}{dx} = \frac{z}{R_1} + \frac{x}{R_1}\frac{\varkappa}{C-A}, \quad \frac{dz}{dx} = \frac{y}{R_1}$$

It follows that

$$y = C_1e^{x/R_1} + C_2e^{-x/R_1} - \frac{R_1\varkappa}{C-A}, \qquad z = C_1e^{x/R_1} - C_2e^{-x/R_1} - \frac{x\varkappa}{C-A}$$

Using equations (4.9) and denoting by u_0, σ_0, n_0, and α_0 the corresponding quantities at the initial instant of time, the solution becomes:

$$u\sigma = (u_0\sigma_0 + r)\cosh(\alpha - \alpha_0) + (n_0\sqrt{k^2 + u_0^2}\,m + r\alpha_0)\sinh(\alpha - \alpha_0) - r$$
$$n\sqrt{k^2 + u^2}\,m = (u_0\sigma_0 + r)\sinh(\alpha - \alpha_0) + (n_0\sqrt{k^2 + u_0^2}\,m + r\alpha_0)\cosh(\alpha - \alpha_0) - r\alpha$$
$$\left(\alpha = \frac{1}{R_1}\sqrt{\frac{C-A}{A}}\sqrt{k^2 + u^2}, \quad r = R_1\frac{\varkappa}{C-A}, \quad m = \sqrt{\frac{C-A}{A}}\right)$$

Now, using the integral of kinetic energies, we get the relations ($f_1(u)$ and $f_2(u)$ are known functions of the coordinate u)

$$\left(\frac{du}{dt}\right)^2 + u^2\left(\frac{dv}{dt}\right)^2 = R_1^2(\sigma^2 + \tau^2) = f_1(u), \qquad u^2\frac{dv}{dt} = R_1 u\sigma = f_2(u)$$

on the basis of which the problem of determining the variables u and v, as functions of time, is reduced to a quadrature. Thus, the question of determining the quantities u, v, σ, r and n is resolved.

The author thanks V.V. Rumiantsev for his assistance.

BIBLIOGRAPHY

1. Woronets P.V. *Uravneniia dvizheniia tverdogo tela, katiashchegosia bez skol'zheniia po nepodvizhnoi ploskosti.* (*The equation of motion of a rigid body, rolling without slipping on a fixed surface*), Kiev, 1903.

2. Woronetz. Uber die Bewegung eines starren Korpers, der ohne Gleitung aut einer beliebigen Flache rollt. *Math. Ann.*, 1910, Vol. 70.

3. Woronetz. Uber die Bewegungsgleichungen eines starren Korpers. *Math. Ann.*, 1911, Vol. 71.

4. Noether. *Uber die rollende Bewegung einer Kugel auf Rotationsflachen.* Munchen, 1909.

5. Chaplygin, S.A. O dvizhenii tiazhelogo tela vrashcheniia na gorizontal'noi ploskosti (The motion of a heavy body of revolution on a horizontal surface) *Sobr. soch.* (*Coll. works*) Vol. 1, Gostekhizdat, 1948.

6. Chaplygin, S.A. O katanii shara na gorizontal'noi ploskosti. (The rolling of a sphere on a horizontal surface) *Sobr. soch.*, (Coll. works) Vol. 1, Gostekhizdat, 1948.

7. Mushtari, H.M., O katanii tiazhelogo tverdogo tela vrashcheniia na nepodvizhnoi gorizontal'noi ploskosti (The rolling of a heavy rigid body of revolution on a fixed horizontal surface). *Matem. sb.* 1932, Vol. 39, No. 1–2.

8. Whittaker, E.T., and Watson, G.N. *Kurs sovremennogo analiza* (*Modern analysis*), Fitzmatgiz, 1961 (Russian translation).

9. Smirnov, V.I., *Kurs vysshei matematiki* (*Course of higher mathematics*), Vol. 3. part 2, Gostekhizdat, 1951.

10. Appel, P. *Teoreticheskaia mekhanika* (*Theoretical mechanics*), Vol. 2, Fizmatgiz, 1961. (Russian translation).

11. Chetaev, N.G. *Ustoichivost' dvizhenii* (*The stability of motion*), Gostekhizdat, 1955.

12. Lur'e, A.I. *Analiticheskaia mekhanika* (*Analytical mechanics*), Fizmatgiz, 1961. (Russian translation).

13. Whittaker, E.T., *Analiticheskaia dinamika* (*Analytical dynamics*) ONTI, 1937 (Russian translation).

Translated by S.A.